Canonical transformation between integrable Hénon-Heiles systems

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A canonical transformation between two known integrable cases of the Hénon-Heiles systems is given and the separation of variables for the corresponding Hamilton-Jacobi equations is discussed.

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The Hénon-Heiles system [1]

$$\dot{q}_1 = p_1, \quad \dot{p}_1 = -2aq_2^2 - bq_1^2, \qquad \dot{q}_2 = p_2, \quad \dot{p}_2 = -2aq_1q_2, \tag{1}$$

has been extensively studied during the past years being one of the "simplest" examples (two particles in a cubic potential) of Hamiltonian system with a mixed phase space structure, i.e., partially ordered and partially chaotic. For generic parameter values a, b the system possesses chaotic orbits and the Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + aq_1q_2^2 - \frac{b}{3}q_1^3, \tag{2}$$

is the only conserved quantity. On the other hand, the system was found to have a second independent integral of motion for the following choices of parameter ratios [2–5]

$$\text{(i)}\ \frac{a}{b} = -1, \quad \text{(ii)}\ \frac{a}{b} = -\frac{1}{6}, \quad \text{(iii)}\ \frac{a}{b} = -\frac{1}{16}. \tag{3}$$

The integrable Hénon-Heiles systems corresponding to the above cases show different degree of complexity. Thus, for example, while the separation of variables problem for case (i) is very simple and it was solved long ago, the same problem appeared to be much more involved for case (iii) and it has been solved only very recently [5]. This has induced to consider the above cases as defining three different cases of integrability for the Hénon-Heiles system.

The aim of the present paper is to show that the Hénon-Heiles systems corresponding to cases (i) and (iii) actually represent the same dynamical system written in different coordinates. More precisely we show the existence of a canonical transformation which maps conserved quantities of case (i) into conserved quantities of case (iii) and vice versa. As an application we construct the separating coordinates for the Hamilton-Jacobi equation for case (iii) by knowing the corresponding ones for case (i) (simply Cartesian coordinates) and applying the transformation.

To construct the canonical transformation, we take advantage of the isomorphism established by Fordy and Gibons [7] between the stationary flows $\dot{\xi} = 0$ of the fifth order partial differential equation (PDE)

$$\xi_t = [\xi_{xxxx} + (8a - 2b)\xi\xi_{xx} - 2(a+b)\xi_x^2 - \tfrac{20}{3}ab\xi^3]_x \tag{4}$$

and the Hénon-Heiles systems corresponding, respectively, to case (i)–(iii). The isomorphism is induced by the transformation

$$\begin{aligned} q_1 &= \xi, \\ p_1 &= \xi_x, \\ p_2^2 &= 2E - \xi_x^2 - \frac{4}{3}b\xi^3 + 2\xi\xi_{xx}, \\ q_2^2 &= \frac{1}{a}(b\xi^2 - \xi_{xx}), \end{aligned} \tag{5}$$

which allows to write the four first order Eqs. (5) with a fixed value of energy (2), as a single fourth order ordinary differential equation (ODE) (and vice versa). In Eqs. (5) the x variable of (4) corresponds to the time variable of the Hénon-Heiles system, i.e., $\xi_x = \dot{q}_1$, $\xi_{xx} = \dot{p}_1$, etc., while the constant E (coming from the space integration in the equation $\dot{\xi} = 0$), simply fixes the energy shell $H = E$ in the Hénon-Heiles systems. Without loss of generality, we fix the values of parameters a, b as

$$\text{case (i)}\ a = 1,\ b = -1, \qquad \text{case (iii)}\ a = \frac{1}{2},\ b = -8. \tag{6}$$

(Note that one of the two parameters, say b, can always be scaled by the transformation $t \to \frac{\tilde{t}}{b}$, $q_i \to b\,\tilde{q}_i$, $p_i \to b^2\,\tilde{p}_i$, $H \to b^4\,\tilde{H}$, so that only the ratio $\gamma = a/b$ is important.) Equation (4), with the above choices of parameters a, b for case (i) and case (iii), becomes [6,7], respectively, the Sawada-Kotera equation:

$$V_t = V_{xxxxx} + 10VV_{xxx} + 20V^2V_x + 10V_xV_{xx}; \tag{7}$$

and the Kaup-Kupershmidt equation:

$$U_t = U_{xxxxx} + 20UU_{xxx} + 80U^2U_x + 50U_xU_{xx}. \tag{8}$$

It is remarkable that Eqs. (7,8) are connected by the following Miura transformations

$$V = u_x - 2u^2 \tag{9}$$

and

$$U = -u_x - u^2, \tag{10}$$

to the same PDE equation

$$u_t = u_{xxxxx} - 10(u_x + 2u^2)u_{xxx} - 10u_{xx}^2 - 80uu_xu_{xx} - 20u_x^3 + 80u^4u_x. \quad (11)$$

This allows to construct a transformation between the Hénon-Heiles system of case (i) and (iii) in the following manner. Let us denote the ξ variable in Eqs. (5) with V (Sawada-Kotera variable) or with U (Kaup-Kupersmidt variable) and the corresponding variables in the Hénon-Heiles system (1) with Q_1, Q_2, P_1, P_2 or q_1, q_2, p_1, p_2, depending on whether they refer to case (i) or case (iii). We take $V = Q_1$, $U = q_1$ in (5) and solve Eq. (10) by taking

$$u = \frac{p_2}{q_2}. \quad (12)$$

Since V and U are both linked to the same Eq. (11) we can substitute (12) in (9) to obtain the Q_1 variable for case (i) in terms of the variables of the Hénon-Heiles system for case (iii), i.e.,

$$Q_1 = -3\left(\frac{p_2}{q_2}\right)^2 - q_1. \quad (13)$$

Substituting this expression in the second equation of (5) and using the equation of motion for case (iii) to eliminate time derivative on variables q, p, we get

$$P_1 = 6\left(\frac{p_2}{q_2}\right)^3 + 6\left(\frac{p_2}{q_2}\right) q_1 - p_1. \quad (14)$$

In a similar manner, we obtain the other transformation equations from the last two equations in (5) as

$$Q_2 = -\frac{\sqrt{F(q,p)}}{q_2^2}, \quad (15)$$

$$P_2 = 2\frac{p_2}{q_2^2}\sqrt{F(q,p)}. \quad (16)$$

In Eqs. (15,16) $F(q,p)$ is just the second integral of motion of the Hénon-Heiles system for case (iii), i.e.,

$$F(q,p) = 9p_1^4 - 6p_1p_2q_2^3 - \frac{q_2^6}{2} + 18p_2^2q_2^2q_1 - 3q_2^4q_1^2. \quad (17)$$

The above transformation maps the Hamiltonian $H(Q,P)$ [put $a = 1, b = -1$ in Eq. (2)] of case (i), into the Hamiltonian $H(q,p)$ [i.e., $a = \frac{1}{2}, b = -8$ in Eq. (2)] of case (iii)

$$H(Q,P) \to H(p,q) \quad (18)$$

and the second integral of motion, $F(Q,P)$, of case (i)

$$F(Q,P) = P_1P_2 + \tfrac{1}{2}(\tfrac{1}{3}Q_2^3 + Q_1^2Q_2) \quad (19)$$

into a function of the second integral of motion of case (iii), i.e.,

$$F(Q,P) \to \frac{\sqrt{F(p,q)}}{6}. \quad (20)$$

Equations (13)–(16), giving the transformation from case (i) to case (iii), can be easily inverted as

$$q_1 = -\frac{3}{4}\left(\frac{P_2}{Q_2}\right)^2 - Q_1, \quad (21)$$

$$p_1 = \frac{3}{2}\left(\frac{P_2}{Q_2}\right)^3 + 3\left(\frac{P_2}{Q_2}\right)Q_1 - P_1, \quad (22)$$

$$q_2 = \sqrt{\frac{-6F(Q,P)}{Q_2}}, \quad (23)$$

$$p_2 = -\frac{1}{2}\frac{P_2}{Q_2}\sqrt{\frac{-6F(Q,P)}{Q_2}}. \quad (24)$$

By rewriting Eqs. (1) in the form

$$\frac{d\mathbf{z}}{dt} = \hat{E}\frac{\partial H}{\partial \mathbf{z}}, \quad (25)$$

where

$$\mathbf{z} = \mathrm{col}\,(q_1, q_2, p_1, p_2),$$

$$\frac{\partial H}{\partial \mathbf{z}} = \mathrm{col}\,(\frac{\partial H}{\partial q_1}, \frac{\partial H}{\partial q_2}, \frac{\partial H}{\partial p_1}, \frac{\partial H}{\partial p_2}),$$

and $\hat{E}$ denotes the (4×4) skew symmetric matrix

$$\hat{E} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-1} & \mathbf{0} \end{pmatrix}, \quad (26)$$

one readily see that the Hénon-Heiles systems of case (i) and (iii) are related by

$$\frac{d\mathbf{z}}{dt} = M\frac{d\mathbf{Z}}{dt}, \qquad \mathbf{Z} = \mathrm{col}\,(Q_1, Q_2, P_1, P_2). \quad (27)$$

In Eq. (27) M denotes the Jacobian matrix $\frac{\partial(q_1,q_2,p_1,p_2)}{\partial(Q_1,Q_2,P_1,P_2)}$ given by

$$M = \begin{pmatrix} -1 & \frac{3}{2}\frac{P_2^2}{Q_2^3} & 0 & -\frac{3}{2}\frac{P_2}{Q_2^3} \\ \frac{-6Q_1Q_2}{\sqrt{-6FQ_2}} & \frac{-2Q_2^2+3P_1P_2}{Q_2\sqrt{-6FQ_2}} & \frac{-3P_2}{\sqrt{-6FQ_2}} & \frac{-2P_1}{\sqrt{-6FQ_2}} \\ 3\frac{P_2}{P_1} & -\frac{3}{2}\frac{P_2(3P_2^2+2Q_1Q_2^2)}{Q_2^4} & -1 & \frac{3}{2}\frac{3P_2^2+2Q_1Q_2^2}{Q_2^3} \\ \frac{3P_2Q_1}{\sqrt{-6FQ_2}} & -\frac{3}{2}\frac{P_2}{Q_2^2}\frac{(3P_1P_2+2Q_1^2Q_2)}{\sqrt{-6FQ_2}} & \frac{3}{2}\frac{P_2^2}{Q_2\sqrt{-6FQ_2}} & \frac{3P_1P_2+6F}{2Q_2\sqrt{-6FQ_2}} \end{pmatrix}. \quad (28)$$

It is remarkable that the matrix M is symplectic, i.e.,

$$M\hat{E}M^T = \hat{E}, \tag{29}$$

this actually proving that the transformation between Q, P and p, q is canonical. Taking into account the isomorphism between cases (i) and (iii) with the stationary flows of, respectively, Sawada-Kotera and Kaup-Koupershmidt equations, this also shows the existence of a transformation between stationary flows of Eqs. (7, 8).

As an application of the above result we can readily obtain the separating variables for the Hamilton-Jacobi equation for case (iii) from those of case (i). Indeed, as well known, $H(P,Q)$ separates in the Cartesian coordinates $\mu_1 = Q_1 + Q_2, \mu_2 = Q_1 - Q_2$. Using the above transformation we then obtain that the corresponding separating coordinates for case (iii) are simply

$$\mu_1 + \mu_2 = -3\left(\frac{p_2}{q_2}\right)^2 - q_1,$$

$$(\mu_1 - \mu_2)^2 = \frac{F(q,p)}{q_2^4}. \tag{30}$$

Note that the variables $\mu_{1,2}$ are elliptic functions and the canonical conjugated variables $\pi_i, \mu_i, i = 1, 2$ are

$$\pi_{1,2} = \frac{d\mu_{1,2}}{dx} = \sqrt{-\frac{4}{3}\mu_{1,2}^3 + E_i \pm F_i}$$

with E_i, F_i given by

$$E_i = \frac{1}{2}(\pi_1^2 + \pi_2^2) + \frac{2}{3}(\mu_1^3 + \mu_2^3),$$

$$F_i = \frac{1}{2}(\pi_1^2 - \pi_2^2) + \frac{2}{3}(\mu_1^3 - \mu_2^3).$$

Equations (30) coincide with those obtained in Ref. [5] in terms of Painlevé analysis performed directly on system (iii).

In conclusion we have shown that the Hénon-Heiles systems corresponding to case (i) and case (iii) are connected by a canonical transformation and, therefore, they represent the same system written in different coordinates. On the other hand, it is known that the Hénon-Heiles system for case (ii) is also isomorphic to the stationary flows of a fifth order PDE [i.e., the second flow of the Korteweg–deVries (KdV) hierarchy] [8]. This raises a natural question: is the Hénon-Heiles system for case (ii) also related to case (i) [or (iii)] by a canonical transformation? Although we cannot exclude this possibility, we remark that such a transformation, if it exists, cannot be constructed by the above method since it appears impossible to map, by Miura transformations, the second equation of the KdV hierarchy and the Sawada-Kotera (or Kaup-Kupersmidth) equation into the same fifth order PDE [i.e., Eq. (11)], as done before.

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